International Journal of Theoretical Physics, Vol. 33, No. 8, 1994

Energy-Momentum Tensor in $N = 1$ Supergravity

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Received November 11, 1993

The classical energy momentum tensor of gravitational waves in $N = 1$ supergravity theory is obtained in analogy with the definition of the same quantity in general relativity.

1. INTRODUCTION

The field equations of general relativity (GR) can be used to define the energy-momentum tensor of the gravitational field by writing the metric in a quasi-Minkowskian coordinate system. This is done in Weinberg (1972) and was originally due to Einstein. On the other hand, the field equations of $N = 1$ supergravity, which can be derived from its Lagrangian, given by van Nieuwenhuizen (1981) and West (1990), by varying the action with respect to the independent tetrad and spinor fields, yield a different result even in matter-free space from those of GR. This is due to the extra fields of the gravitinos as well as to the torsion introduced in space-time by their presence. The aim of this paper is to explore the possibility of setting up a similar formulation of classical gravitational waves in $N = 1$ supergravity from its field equations in free space to the one existing in GR and hence to define the energy-momentum tensor of the same field.

2. FIELD EQUATIONS OF $N = 1$ SUPERGRAVITY

As is well known (van Nieuwenhuizen, 1981), the total Lagrangian of $N = 1$ supergravity consists of a bosonic and a fermionic part,

$$L = L^{(2)} + L^{(3/2)} \tag{1}$$

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where

$$L^{(2)} = -(1/2\kappa^2)ee^{a\nu}e^{b\mu}R_{\mu\nu ab}(\omega) \tag{2}$$

$$L^{(3/2)} = (1/2)\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\nu\gamma_5 e^m_\mu\gamma_m\left(\frac{\partial}{\partial x^\rho}+\frac{1}{2}\,\omega^{pq}_\rho\sigma_{pq}\right)\psi_\sigma \tag{3}$$

and

$$R_{\mu\nu}{}^{mn}(\omega) = \partial_\mu\omega^{mn}_\nu - \partial_\nu\omega^{mn}_\mu + \omega^{me}_\mu\,\omega^n_{\nu e} - \omega^{me}_\nu\omega^n_{\mu e} \tag{4}$$

We have used the fact that $\gamma_\nu = e^m_\nu\gamma_m$. The variation of action $I = \int L\,d^4x$ with respect to each of the fields $\bar{\psi}_\mu$, ψ_μ, ω^{mn}_μ, and the tetrad e^m_μ all regarded as independent must vanish, that is,

$$\frac{\delta I}{\delta e} = \frac{\delta I}{\delta\psi} = \frac{\delta I}{\delta\bar{\psi}} = \frac{\delta I}{\delta\omega} = 0 \tag{5}$$

Of these the last condition has already been worked out in van Nieuwenhuizen (1981) and ω^{mn}_μ has been eliminated as an independent field, yielding

$$\omega_{\mu mn}(e,\psi) = \omega_{\mu mn}(e) + \frac{\kappa^2}{4}(\bar{\psi}_\mu\gamma_m\psi_n - \bar{\psi}_\mu\gamma_n\psi_m + \bar{\psi}_m\gamma_\mu\psi_n) \tag{6}$$

where

$$\omega_{\mu mn}(e) = \frac{1}{2}\,e^\nu_m(\partial_\mu e_{n\nu} - \partial_\nu e_{n\mu}) - \frac{1}{2}\,e^\nu_n(\partial_\mu e_{m\nu} - \partial_\nu e_{m\mu})$$
$$-\frac{1}{2}\,e^\rho_m e^\sigma_n(\partial_\rho e_{c\sigma} - \partial_\sigma e_{c\rho})e^c_\mu \tag{7}$$

Thus $\omega_{\mu mn}$ is still antisymmetric in the indices m and n because the term $\bar{\psi}_m\gamma_\mu\psi_n$ can be shown to be antisymmetric (van Nieuwenhuizen, 1981). The conditions $\delta I/\delta e = 0$, $\delta I/\delta\bar{\psi} = 0$, and $\delta I/\delta\psi = 0$ yield, respectively, the equations

$$\frac{1}{2}\,e^n_\lambda R - e^\mu_b R_{\mu\lambda}{}^{nb} - \frac{k^2}{2e}\,\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\nu\gamma_5\gamma_m\left(\frac{\partial}{\partial x^\rho}+\frac{1}{2}\,\omega^{pq}_\rho\sigma_{pq}\right)\psi_\sigma e^n_\mu e^m_\lambda = 0 \tag{8}$$

$$\frac{1}{2}\,\epsilon^{\mu\nu\rho\sigma}\gamma_5 e^m_\mu\gamma_m\left(\frac{\partial}{\partial x^\rho}+\frac{1}{2}\,\omega^{pq}_\rho\sigma_{pq}\right)\psi_\sigma = 0 \tag{9}$$

and

$$\frac{1}{2}\,\epsilon^{\mu\nu\rho\sigma}\left[\frac{1}{2}\,\bar{\psi}_\nu\gamma_5 e^m_\mu\gamma_m\omega^{pq}_\rho\sigma_{pq} - \frac{\partial\bar{\psi}_\nu}{\partial x^\rho}\gamma_5 e^m_\mu\gamma_m - \bar{\psi}_\nu\gamma_5\frac{\partial e^m_\mu}{\partial x^\rho}\gamma_m\right] = 0 \tag{10}$$

Each of equations (8) and (10) consists of a set of four equations corresponding to the values of the indices ν and σ, respectively. Thus the components of the vectorial spinor ψ_σ (consisting of two independent real vectorial spinors) can be evaluated in terms of the tetrad from these eight matrix differential equations. On substituting these expressions along with that of ω_μ^{mn} from equations (6) and (7) into equation (8), we finally get the free-space field equations of $N=1$ supergravity expressed in terms of only the bosonic tetrads e_μ^n. The process is cumbersome.

3. LINEARIZED FIELD EQUATIONS OF SUPERGRAVITY AND THE ENERGY-MOMENTUM TENSOR OF GRAVITATION

The free linearized field equations derived in Section 1.13 of van Nieuwenhuizen (1981) show that the bosonic tetrad field equations and the gravitino equations are uncoupled. This is unlike the full field equations (8)–(10) along with equation (6) for the spin connection, which are all coupled and, as noted above, allow us to eliminate the gravitino field and express all the components of the field equations in terms of only those of the tetrad. We thus believe that only one of the sets of the linearized field equations, preferably that of the tetrad, will suffice to express the energy-momentum tensor of gravity waves, as the gravitino can be evaluated in terms of the graviton field. The energy-momentum tensor of the gravitational waves in GR is defined by equation (7.6.4) of Weinberg (1972) as

$$t_{\mu\kappa} = (1/8\pi G)[R_{\mu\kappa} - (1/2)g_{\mu\kappa}R_\lambda^\lambda - R_{\mu\kappa}{}^{(1)} + \tfrac{1}{2}\eta_{\mu\kappa}R_\lambda^{\lambda(1)}] \tag{11}$$

The quantity analogous to $R_{\mu\kappa}{}^{(1)} - \frac{1}{2}\eta_{\mu\kappa}R_\lambda^{\lambda(1)}$ can be determined for supergravity by comparing equation (3) of Section 1.13 of van Nieuwenhuizen (1981) with equation (10.1.4) of Weinberg (1972). The full energy-momentum tensor of the field is thus

$$t_\lambda{}^\mu = -(1/2\kappa^2)\bigg[\delta_\lambda^\mu R - 2e_n^\mu e_b^\sigma R_{\sigma\lambda}^{nb} - \frac{\kappa^2}{e}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\nu\gamma_5\gamma_m \times\left(\frac{\partial}{\delta x^\rho} + \frac{1}{2}\omega_\rho^{pq}\sigma_{pq}\right)\psi_\sigma e_\lambda^m + \Box\chi_\lambda^\mu - \partial_\lambda\chi^\mu - \partial^\mu\chi_\lambda + \delta_\lambda^\mu\partial_\alpha\chi^\alpha\bigg] \tag{12}$$

where χ_λ^μ is given by equation (3) of Section 1.13 of van Nieuwenhuizen (1981) and $R_{\mu\nu}^{mn}$, ψ_σ, and ω_μ^{mn} are to be substituted from the respective equations derived earlier.

The energy-momentum tensor so defined is asymmetric in indices μ and λ and this is to be expected because of the presence of torsion. In ordinary Einstein–Cartan theory (Hehl *et al.*, 1976) it is seen that the Ricci and Einstein tensors are asymmetric and an energy-momentum tensor constructed out of the prescription followed in this paper will also be inherently asymmetric.

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